Math and Nature

* The dawn of the scientific age coincided with the realization that nature can be described mathematically. As stated by Galileo in 1623, the entire universe is written in the language of mathematics. Since this time, quantitative analysis of natural phenomena has been at the heart of scientific inquiry. Owing to this relationship between natural phenomena and its description via mathematics, nature provides a tangible context for math that can be used to invite students into math instruction. Establishing context is essential for developing student interest and comprehension of a topic, and for promoting long-term problem solving skills.

The Importance of Context

* In writing, context is the setting in which a story occurs. In many ways the setting gives meaning to the details of the story and is therefore essential to the development of the plot. As educators we are, in a sense, storytellers that can use context to invite students into the material of our classes. Because of this, we believe that many of the components of good story-telling can be used to improve student engagement and knowledge in the classroom.
	+ A lack of context is one of the more common complaints about math education because students are always wondering why a particular mathematical topic is relevant. Context establishes relevance, thereby increasing student interest.
* Pedagogical research has shown that creating context allows students to relate current material to their prior experiences, thereby facilitating experiential and associative learning. This was described by Merrill (2002):
	+ First Principles of Instruction:
		- The demonstration principle: Learning is promoted when learners observe a demonstration.
		- The activation principle: Learning is promoted when learners activate prior knowledge or experience.
		- The application principle: Learning is promoted when learners apply the new knowledge.
		- The task-centered principle: Learning is promoted when learners engage in a task-centered instructional strategy.
		- The integration principle: Learning is promoted when learners integrate their new knowledge into their everyday world.
	+ The demonstration of new knowledge within a relevant context activates students’ prior knowledge of the subject, which engages them in the material at hand. Once engaged, students are given the opportunity to apply this new knowledge in a task-centered (problem-solving) manner. The association of new knowledge with prior knowledge reinforces the material in memory, thereby allowing learned concepts to be applied to new experiences. Integration of knowledge into new experiences is the essence of problem-solving.

Geometry and Biology

* The use of nature as a context for math is extremely evident with regard to biology and geometry. Biological structures vary greatly in their geometry. Given that differences in the geometry of biological structures affect the function of those structures, biology provides a mechanism for geometric problem solving and for illustrating the consequences of geometric variability.
	+ The consequences of geometric variability establish the functional aspect of biology, which is what really invites students into the problem. Given the extensive diversity of organisms, their function can provide context for most any math topic.

Shell Morphology

* The main idea behind this lesson is that of “morphospace”, the theoretical range of possible geometries in organisms. This can easily be envisioned as a circle at the origin of a plot that is morphed in space along X and Y axes. The horizontal radius of the circle increases on the X axis and the vertical radius of the circle increases on the Y axis. All coordinates in X-Y space therefore have an oval of varying dimension associated with them. The purpose of this lesson will be to explore the morphospace of shells in order to determine which portions of it are occupied by organisms, which portions are not, and why these differences exist.
	+ Portions of morphospace may not be occupied not occupied for one of three reasons:
		- Evolution has not produced all possible geometries because it is a random process.
		- Extinction has eliminated certain geometries that once existed
		- Functional constraints exist on morphology such that certain portions of morphospace are disadvantageous. Therefore, any organisms that possessed these characteristics would have been unsuccessful and the disadvantageous trait would not have been passed on to subsequent generations.
			* For example, there are no birds with short stubby wings because short stubby wings cannot be used to fly.
	+ The portions of morphospace that are occupied are filled with species to varying extents, which can be envisioned like a topographic map. Those areas bounded by the most circles are the “adaptive peaks” in the morphospace, presumably because those geometries are the most advantageous. Kauffman (1995) described life as a “high-country adventure” given the fact that most biodiversity is clustered around these peaks and the remaining valleys of morphspace are often unoccupied.
		- Convergent evolution, the process through which organisms independently arrive at a similar solution to a particular problem, is classic evidence of this. For example, birds, bats, and pterodactyls all have wings of similar geometry because that is the only way to travel through the air, yet these three groups of organisms independently arrived at the same adaptive peak during the course of evolutionary history.
* Regarding the evolution of shell geometry, spiral forms have convergently evolved on numerous occasions in organisms ranging from unicellular foraminiferans to large gastropods like queen conch. A tremendous amount of research on shell geometry has focused on ammonites, an extinct group of cephalopods related to the modern-day nautilus, squid, and octopi, which had an amazing array of shell geometries. This group once had several hundred species and far out-numbered the only other shelled cephalopods, the nautilids.
	+ Ammonite morphospace is represented by a logarithmic spiral which creates a twisted cone. The terms describing the geometry of these cones include:
		- Whorl expansion rate (W) – the rate at which the open end of the cone increases in diameter relative to the tip of the cone.
		- Distance from the coiling axis (D) – the distance that the open end of the cone extends outward from the origin of the spiral which each whorl.
		- Translation rate (T) – the rate at which the cone extends downward along the coiling axis.
* Data from ~400 species indicates that the region of morphospace most commonly occupied by ammonites is that where the whorls of the cone overlap with each successive turn, and that fewer ammonites occupy the region of morphospace with non-overlapping whorls. The purpose of this exercise is to figure out why this difference exists.
	+ Evidence suggests that a functional constraint exists on shells with non-overlapping whorls because non-overlapping whorls have greater collective surface area and allow water to flow between the whorls, resulting in greater drag in the water and lower swimming efficiency. Conversely, shells with overlapping whorls have a continuous surface with less surface area, which experiences less drag in the water and has higher swimming efficiency.
		- Interestingly, the nautilids used to have shells that occupied the less favorable portion of the morphospace, and when the ammonites went extinct, the nautilids shifted to occupy a more favorable portion of the morphospace. This corroborates that idea that similar organisms are competing to occupy the preferred portions of morphospace via their geometries.
			* The ammonites are believed to have gone extinct at the end of the Cretaceous Period (approximately 65 million years ago) in the same mass extinction event (meteor collision off of the Yucatan Peninsula of Mexico) that wiped out the non-bird dinosaurs and many other organisms.
* Repeated convergent evolution
	+ On at least three separate occasions there have been major changes in ammonite shell geometry with respect to changes in sea level. When sea level is low and there are only deep, open oceans (no shallow oceans on the continental shelves), ammonite shells have numerous whorls and extensive ornamentation (irregularities of the shell surface). However, when sea level rises and shallow seas form on the margins of continents, ammonites diversify to have shells with fewer, larger whorls and less ornamentation; when sea level falls again simpler, less ornamented species go extinct.
		- The rationale for these changes is that shells with less ornamentation have less surface area and therefore experience less drag in shallow seas, in which there is extensive water movement (deep seas are relatively still and have less water movement). Alternatively, the presence of numerous whorls in deep sea ammonites provides greater resistance against the water pressure of the deep sea because each whorl is made of calcium, which is very sturdy (like our bones!)

Ammonite Shell Models

* The ultimate question to be answered by this lesson regards the manner in which shell surface area and volume differ among ammonites with overlapping and non-overlapping whorls? The procedure consists of creating two identical cones from modeling clay, twisting them into shell models with non-overlapping and overlapping whorls, and determining differences in surface area and volume between these models.
* Procedure
	+ Use clay to create two shell models of equal size.
	+ Measure their height and radius.
	+ Twist one of the cones into a model with non-overlapping whorls and the other into a model with overlapping whorls.
	+ Measure the height and radius of the model with overlapping whorls.
	+ Calculate the volume of the space where the organism lives using the measurements of the original cones.
		- $Volume=\frac{1}{3}πr^{2}h$
		- Sample data:
			* $r=0.50 cm$
			* $h=18.00 cm$
			* $Volume =4.71 cm^{3}$
	+ Calculate the surface area of both cones. Note that the surface area of the cone with non-overlapping whorls will be the same as the surface area of the original cone.
		- $Surface Area=πr\left(r+\sqrt{h^{2}+r^{2}}\right)$
		- Sample data:
			* Non-overlapping whorls
				+ $r=0.50 cm, h=18.00 cm$
				+ $Surface area =29.07 cm^{2}$
			* Overlapping whorls
				+ $r=1.25 cm, h=3.00 cm$
				+ $Surface area =17.67 cm^{2}$
	+ Calculate the surface area-to-volume ratio for each shell model.
		- Sample data:
			* Non-overlapping whorls
				+ $\frac{Surface area}{Volume}=6.17$
			* Overlapping whorls
				+ $\frac{Surface area}{Volume}=3.75$
	+ Determine which cone model has better swimming efficiency (i.e., less drag due to surface area).
		- The model with overlapping whorls has less drag and better swimming efficiency due to its lower surface area to volume ratio, which explains why the region of morphospace with overlapping whorls is the most heavily occupied by ammonites.
* Additional work
	+ Comparisons of deep water and shallow water shell types can be performed by simulating the ornamentation of deep sea species with geometric objects (e.g., cones, pyramids) added to the surface of the shell. Shallow water forms do not have these ornaments, which minimizes surface area and drag. This attribute is important in shallow oceans, which have much more fluid motion than the deep sea.

References

* Bayer, U. and McGhee, G.R. (1984). Iterative evolution of Middle Jurassic ammonite faunas. Lethaia. 17: 1-16.
* Kauffman, S. (1995). At Home in the Universe: The Search for Laws of Self-Organization and Complexity. Oxford University Press.
* McGhee, G.R. (2006). The Geometry of Evolution. Cambridge University Press.
* Raup, D. M. (1966). Geometric analysis of shell coiling: general problems. Journal of Paleontology. 40: 1178 – 1190.
* Raup, D. M. (1967). Geometric analysis of shell coiling: coiling in ammonoids. Journal of Paleontology. 41: 42-65.